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Technical Bulletin

Mathematics of Frequency Optimization for Maximum Transfer Power

Magnetic components can be made as small as possible for a specified output power if they deliver maximum core transfer power. One of the optimizations for achieving minimum core size is to operate the core at the switching frequency, f_s, that has acceptable power loss for converters operating deep in continuous-current mode (CCM). The linear equation for average transfer power through a *multi-winding transformer* or *coupled inductor* (or *transductor*) is

$$\overline{P} = \Delta W_L \cdot f_s = [\Delta B \cdot \overline{H} \cdot V] \cdot f_s = [(2 \cdot \hat{B}_{\sim}) \cdot \overline{H} \cdot V] \cdot f_s$$

where the energy transferred per cycle = ΔW_L, magnetic field density ripple (~) amplitude (^) = $\hat{B}_{\sim} = \Delta B/2$, average (¯) field intensity = $\overline{H}$, and core volume = V. Linearity is assumed for magnetic operation whenever the *small-ripple approximation* ($dB \approx \Delta B << \overline{B}$) applies, such as converters operating deep in CCM. They have large average current and small Δi, with a *ripple factor*, $\gamma = (\Delta i / 2) / \bar{i} = \hat{i}_{\sim} / I << 1$. (The boundary between CCM and DCM is at $\gamma = 1$, where CCM is $\gamma \leq 1$.) The average on-time circuit current, I, corresponds to $\overline{H}$ in the $\overline{P}$ equation, and is a static value – a constant. For a given core, the geometry is fixed; thus, V is constant and only $\hat{B}_{\sim}(f_s)$ varies in $\overline{P}$ with frequency.

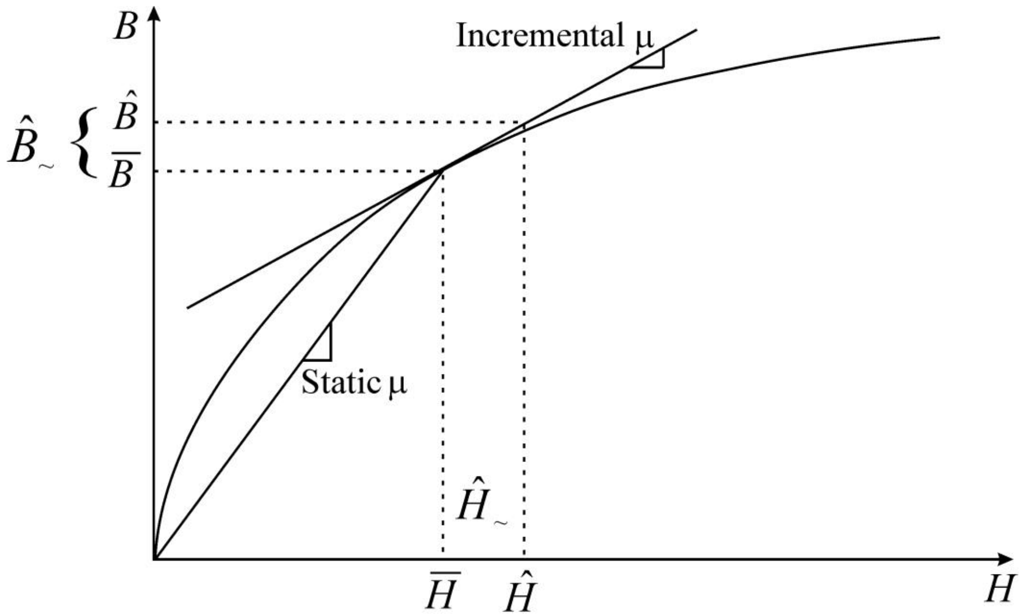
Magnetics Linearization

The field intensity ripple (~) amplitude (^), $\hat{H}_{\sim}$ also varies linearly with current ripple in the circuit and is usually kept constant by controlling the peak on-time current. Then the *incremental permeability*, μ, at the op-pt, $\overline{H}_0$, is

$$\textit{Incremental } \mu = \frac{dB}{dH}, \; \overline{H} = \overline{H}_0$$

Graphically, incremental μ at an operating point of $\overline{H} = \overline{H}_0$ is shown below. The static μ is the slope of the line from the origin to the magnetic operating point of the core at $(\overline{B}, \overline{H})$ and is not the same as the incremental μ, which is the slope of the line tangent to the $B(H)$ curve at the operating point and is the derivative, $dB/dH \approx \Delta B/\Delta H$. For nonlinear functions such as $B(H)$, static and incremental μ are not the same. Variation of B for small variations of H around $\overline{H}$ are linearized by moving along the tangent line, and if ΔH is small, is approximately the same as moving along the $B(H)$ curve itself, resulting in an accurate approximation of magnetic behavior.

For linear components, such as resistors, static and incremental (or *small-signal*) parameters, R and r, are the same; $r = R$. But for nonlinear semiconductors such as Si p-n junctions, a voltage drop of 0.65 V at 1 mA has a static $R = 0.65$ V/1 mA = 650 Ω, whereas a small change in current around 1 mA produces a small change in voltage determined by $r = \Delta v/\Delta i = 26\ \text{mV}/I \approx 26\ \Omega$, an incremental resistance that is much less than the static resistance. Magnetic cores as nonlinear devices can be linearized in the same way as p-n junctions.



Maximum Transfer-Power Conditions

Per-cycle transfer power occurs at a rate of f_s and output power increases proportionally to frequency for constant $\hat{B}_{\sim}$. However, as $\hat{H}_{\sim}$ is held constant, $\mu(f_s)$ decreases with f_s, causing ΔW_L to decrease with frequency. With a constant H waveform, $\hat{B}_{\sim} = \mu(f_s) \cdot \hat{H}_{\sim}$ also decreases with f_s.

The transfer-power equation can be regrouped into constant and frequency-dependent factors:

$$\overline{P} = \Delta W_L \cdot f_s = [2 \cdot \overline{H} \cdot V] \cdot [\hat{B}_{\sim}(f_s) \cdot f_s] = \text{constant} \cdot [\hat{B}_{\sim} \cdot f_s]$$

As f_s increases, $\hat{B}_{\sim}(f_s)$ of magnetic materials decreases. Maximum $\overline{P}(f_s)$ is found by setting the derivative of the transfer power to zero and solving:

$$\frac{d\overline{P}(f_s)}{df_s} = \text{constant} \cdot \frac{d}{df_s}[\hat{B}_{\sim} \cdot f_s] = 0 \;\Rightarrow\; \frac{d}{df_s}[\hat{B}_{\sim} \cdot f_s] = 0 \text{ , constant} \neq 0$$

Then differentiating, the maximum (or constant) power occurs under the condition that

$$\hat{B}_{\sim} + f_s \cdot \frac{d\hat{B}_{\sim}}{df_s} = 0 \;\Rightarrow\; \frac{d\hat{B}_{\sim}}{\hat{B}_{\sim}} = -\frac{df_s}{f_s} \;\Rightarrow\; \frac{\dfrac{d\hat{B}_{\sim}}{\hat{B}_{\sim}}}{\dfrac{df_s}{f_s}} = -1$$

The equation shows that maximum power occurs whenever the fractional decrease in B ripple amplitude equals the fractional increase in the frequency. The two changes cancel and $\overline{P}$ remains nearly constant around the maximum point. Integrate both sides of the above (center) differential equation, and the result is

$$\ln \hat{B}_{\sim} = -\ln f_s + C$$

where C is the arbitrary constant of integration. Choose the operating point, (f_{s0}, $\hat{B}_{\sim 0}$) to determine C. Then

$$C = \ln \hat{B}_{\sim 0} + \ln f_{s0}$$

Substituting and rearranging,

$$\ln\frac{\hat{B}_\sim}{\hat{B}_{\sim 0}} = -\ln\frac{f_s}{f_{s0}} = \ln\frac{f_{s0}}{f_s} \Rightarrow \frac{\hat{B}_\sim}{\hat{B}_{\sim 0}} = \frac{f_{s0}}{f_s} \Rightarrow \boxed{\hat{B}_\sim = \frac{f_{s0}}{f_s}\cdot\hat{B}_{\sim 0}}$$

When this expression for $\hat{B}_\sim$ at maximum transfer power is substituted back into the transfer-power equation, then

$$\overline{P} = [2\cdot\overline{H}\cdot V]\cdot[\hat{B}_{\sim 0}\cdot f_{s0}] \Rightarrow \overline{P}/\overline{P}_0 = 1$$

The operating point, (f_{s0}, $\hat{B}_{\sim 0}$) is at maximum transfer power under this condition.

Core Power Loss Density Exponents

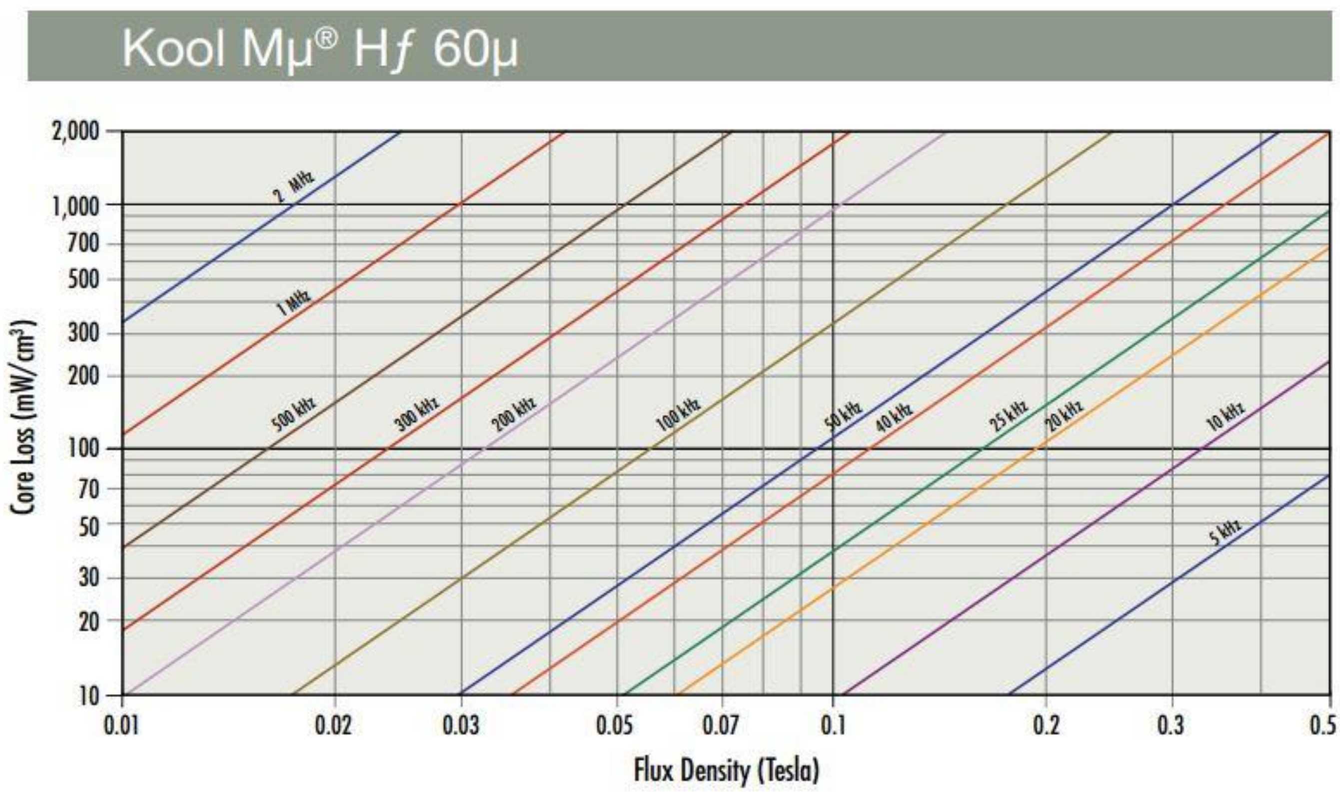
Average core power loss density, $\overline{p}_c$, also imposes a limit on f_s. The *generalized Steinmetz equation*, normalized to an operating point at $\overline{p}_{c0}(f_{s0}, \hat{B}_{\sim 0})$ is

$$\frac{\overline{p}_c}{\overline{p}_{c0}} = \left(\frac{f_s}{f_{s0}}\right)^{\alpha}\cdot\left(\frac{\hat{B}_\sim}{\hat{B}_{\sim 0}}\right)^{\beta}$$

where exponents α and β depend on the material and are empirically determined. (Normalization eliminates a constant in the equation by using unitless ratios and removes the messiness of raising parameters with units to non-integer powers.) The "classical" values for the exponents are $\alpha = 2$ and $\beta = 2$, but they vary with material and frequency. For typical values, P-material ferrites have $\alpha \approx 1.36$ and $\beta \approx 2.62$, with $\overline{p}_{c0}$ at operating point,

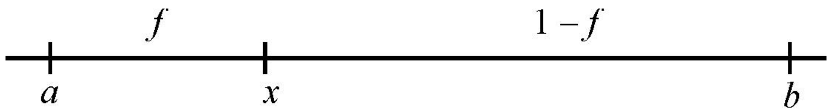
$$\overline{p}_{c0}(f_{s0}, \hat{B}_{\sim 0}) = \overline{p}_{c0}\,(100\text{ kHz, }11.5\text{ mT}) = 100\text{ mW/cm}^3$$

To find the exponent values, β is the slope of the log-log plots of $\overline{p}_c(\hat{B}_\sim)$, as graphed below for Magnetics Kool Mμ H*f* (KMHF) with relative permeability $\mu_r = 60$ (60μ) and with f_s held constant as the plot parameter. On the 100 kHz plot is the value $\overline{p}_c(\hat{B}_\sim)$ = 100 mW/cm³(55 mT).



Interpolation of Log Scales

A horizontal log axis is shown below.



Along the log scale, linear interpolation of a value at x between graph values of a and b is the fraction of linear distance, f between a and x and $1 - f$ between x and b. The linear fraction of distance of log(x) from log(a) between log(b) and log(a) is

$$f = \frac{\log x - \log a}{\log b - \log a} = \frac{\log\left(\frac{x}{a}\right)}{\log\left(\frac{b}{a}\right)} = \log_{(b/a)}(x/a)$$

To find the scale value of x, solve for x;

$$\boxed{x = a \cdot \left(\frac{b}{a}\right)^{f}}$$

The greatest need for interpolation is between 1 and 2 (or powers of ten thereof) and the rule can be applied:

$$x \approx 1 + f\,,\ f \in [0.1, 0.2], [0.9, 1.0]$$

$$x \approx 1 + (f - 0.1),\ f \in [0.2, 0.9]$$

For example, for $f = 0.5$, $x = 1 + (0.5 - 0.1) = 1.4$. The more accurate value is 1.41. For $a = 0.04$ T, $b = 0.05$ T, and $f = 0.44$, then at 100 mW/cm^3, $x = (0.04\text{ T})\cdot(0.05/0.04)^{0.44} = 44$ mT.

The α exponent is found from the graph around the 100 kHz operating point by holding $\Delta\hat{B}_\sim$ constant and finding

$$\alpha = \frac{\Delta\bar{p}_c}{\Delta f_s}\ ,\ \Delta\hat{B} = 0\,\text{mT}$$

The operating point $\hat{B}_{\sim 0} \approx 55$ mT (0.055 T) and two values of $\bar{p}_c$ an octave on each side of the 100 kHz plot give

$$\alpha = \frac{\Delta\bar{p}_c}{\Delta f_s} = \frac{\log(300\text{ mW/cm}^3) - \log(40\text{ mW/cm}^3)}{\log(200\text{ kHz}) - \log(50\text{ kHz})} = \frac{\log(7.5)}{\log(4)} = 1.45\,,\ \hat{B}_\sim = 55\,\text{mT}$$

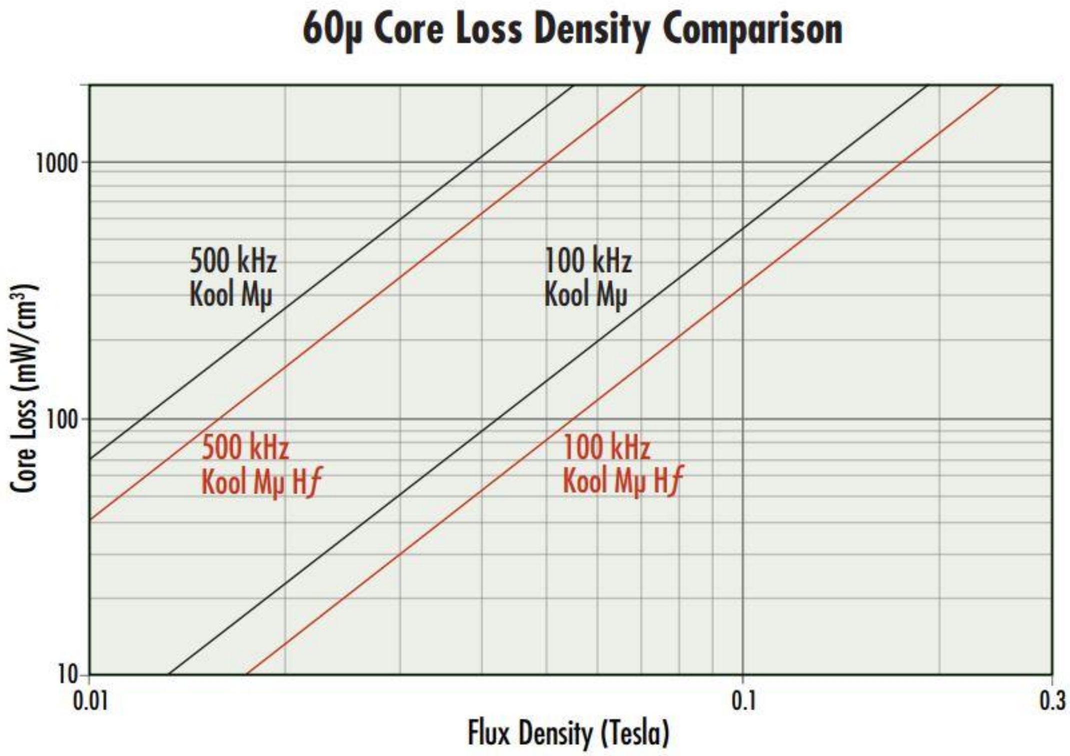
Shown below for comparison are power loss plots of both Kool Mμ (KM) and Kool Mμ Hf (KMHF) alloys for $\mu_r = 60$ at $f_s = 100$ kHz and 500 kHz. Values from the power loss graph are:

Kool Mμ (KM): $\bar{p}_c$ = 100 mW/cm^3, at $\hat{B}_\sim$ = 42.2 mT, f_s = 100 kHz and $\hat{B}_\sim$ = 12.1 mT, f_s = 500 kHz

Kool Mμ Hf (KMHF): $\bar{p}_c$ = 100 mW/cm^3, at $\hat{B}_\sim$ = 55.4 mT, f_s = 100 kHz, and $\hat{B}_\sim$ = 15.8 mT, f_s = 500 kHz

At the same power loss and at 100 kHz, KMHF $\hat{B}_\sim$ is about 31 % higher than KM which corresponds (from the $\bar{P}$ equation) to 31% greater transfer power through the core. At 500 kHz, the ratio of KMHF/KM transfer power advantage is maintained at 31%.

From the graph below, the values of α are derived for both KM and KMHF materials, from the following values read from the graph. The operating point is ($\hat{B}_{\sim 0}$, f_s) = (50 mT, 223.6 kHz), where $f_{s0} = \sqrt{(100\,\text{kHz})\cdot(500\,\text{kHz})}$. The KMHF α value agrees with the previously calculated value, showing that there is no significant variation in α with frequency for these materials.



Quantity	f_s, kHz	KM, mW/cm³	KMHF, mW/cm³
$\overline{p}_c$(50 mT)	100	141	82
$\overline{p}_c$(50 mT)	500	1652	950
ratio	5	11.72	11.59
log (ratio)	0.699	1.069	1.064
α		1.53	1.52

The KMHF β exponent is the plot slope. Around the operating point of $(f_{s0},\ \hat{B}_{\sim 0})$ = (100 kHz, 50 mT), it is

$$\beta = \frac{\log(340\text{ mW/cm}^3) - \log(30\text{ mW/cm}^3)}{\log(100\text{ mT}) - \log(30\text{ mT})} = \frac{\log(340/30)}{\log(100/30)} = \frac{1.054}{0.523} = 2.02$$

Because both the KM and KMHF plots appear parallel, β is the same for both and also appears to not vary over a range of f_s in that all the plots are parallel. For both, $\beta \approx 2$, its "classical" value.

Finally, the power loss density equation as expressed for KMHF around op-pt, $(f_{s0},\ \hat{B}_{\sim 0})$ = (100 kHz, 50 mT) is

$$\frac{\overline{p}_c}{(100\text{ mW/cm}^3)} = \left(\frac{f_s}{100\text{ kHz}}\right)^{1.5} \cdot \left(\frac{\hat{B}_{\sim}}{55\text{ mT}}\right)^2$$

For KM cores,

$$\frac{\overline{p}_c}{(100\text{ mW/cm}^3)} = \left(\frac{f_s}{100\text{ kHz}}\right)^{1.5} \cdot \left(\frac{\hat{B}_{\sim}}{42\text{ mT}}\right)^2$$

These Steinmetz equations of KM and KMHF core materials differ only in that KMHF field density is 31% higher at the same power loss and frequency as KM material.

Maximum Power Loss Conditions

Maximum power loss with f_s is derived by setting the differentiated Steinmetz equation to zero;

$$\frac{d}{df_s}\left(\frac{\overline{p}_c}{\overline{p}_{c0}}\right) = \frac{1}{f_{s0}^{\alpha}\cdot\hat{B}_{\sim0}^{\beta}}\cdot\left[\alpha\cdot f_s^{\alpha-1}\cdot\hat{B}_{\sim}^{\beta} + f_s^{\alpha}\cdot\beta\cdot\hat{B}_{\sim}^{\beta-1}\cdot\frac{d\hat{B}_{\sim}}{df_s}\right]$$

$$= \frac{f_s^{\alpha-1}\cdot\hat{B}_{\sim}^{\beta-1}}{f_{s0}^{\alpha}\cdot\hat{B}_{\sim0}^{\beta}}\cdot\left[\alpha\cdot\hat{B}_{\sim} + f_s\cdot\beta\cdot\frac{d\hat{B}_{\sim}}{df_s}\right] = 0$$

Solving for the condition for constant loss, it is the fractional change in B to the fractional change in f_s at constant core power loss density:

$$\boxed{\frac{d\hat{B}_{\sim}/\hat{B}_{\sim}}{df_s/f_s} = -\frac{\alpha}{\beta}\ ,\ \Delta\overline{p}_c = 0\ \text{mW/cm}^3}$$

At maximum power loss, the $\overline{p}_c(f_s)$ curve peaks, and at the peak the change in $\overline{p}_c$ with f_s is minimum, i.e. the point where the slope of the tangent to $\overline{p}_c$ is zero and hence constant. For the classic values of $\alpha = 2$, $\beta = 2$, then

$$\frac{d\hat{B}_{\sim}/\hat{B}_{\sim}}{df_s/f_s} = -1$$

Constant power loss occurs under the same condition as constant transfer power. Consequently, they are independent of f_s whenever $\alpha/\beta = 1$.

When the constant-loss equation above is solved, the constraint on constant power loss is that

$$\boxed{\frac{\hat{B}_{\sim}}{\hat{B}_{\sim0}} = \left(\frac{f_s}{f_{s0}}\right)^{-\alpha/\beta}\ ,\ \Delta\overline{p}_c = 0\ \text{mW/cm}^3}$$

When substituted into the Steinmetz equation, $(f_s/f_{s0})^0 = 1$ results, and $\overline{p}_c = \overline{p}_{c0}$.

The constant power loss constraint can be substituted into the transfer-power equation normalized to $\overline{P}_0(\hat{B}_{\sim0}, f_{s0})$:

$$\frac{\overline{P}}{\overline{P}_0} = \left(\frac{\hat{B}_{\sim}}{\hat{B}_{\sim0}}\right)\cdot\left(\frac{f_s}{f_{s0}}\right) = \left(\frac{f_s}{f_{s0}}\right)^{-\alpha/\beta}\cdot\left(\frac{f_s}{f_{s0}}\right) \Rightarrow \boxed{\frac{\overline{P}}{\overline{P}_0} = \left(\frac{f_s}{f_{s0}}\right)^{1-\frac{\alpha}{\beta}}\ ,\ \frac{\overline{p}_c}{\overline{p}_{c0}}\ \text{constant}}$$

This is the transfer power as a function of f_s with constant magnetic power loss. The closer α/β is to 1, the less dependent the transfer power is on frequency. For materials with $\alpha/\beta < 1$, transfer power rises with frequency at constant power loss. For KM and KMHF materials, $\alpha/\beta = 1.5/2 = 0.75 < 1$. The KMHF power loss plots show that the slope β does not decrease at the maximum frequency plot of 2 MHz.

Similarly, if the constant transfer power condition is substituted into the power loss equation, then

$$\boxed{\frac{\overline{p}_c}{\overline{p}_{c0}} = \left(\frac{f_s}{f_{s0}}\right)^{\alpha-\beta}\ ,\ \frac{\overline{P}}{\overline{P}_0}\ \text{constant}}$$

For $\alpha < \beta$, the exponent is negative and power loss (along with $\hat{B}_{\sim}$) decreases with f_s under constant transfer power. For $\alpha = \beta$, power loss is independent of (and constant with) frequency.

Consequently, choice of core material is optimized whenever transfer power relative to power loss is maximized, and this occurs for a minimum α/β.

When considering maximum operating frequency for a material, the $\mu(f_s)$ curve must also be taken into account, as inductance diminishes with frequency. The μ-related frequency parameter is f_μ, the frequency at which (the real or dissipative component of) μ decreases to 90% of its quasistatic value. As $\mu(f_s)$ decreases, so does field inductance, $\mathcal{L}$

and transfer power. It is not necessarily the case that a material cannot be useful above f_μ. Impedance is proportional with frequency, and with the result that peak impedance for a wound core occurs at a much higher frequency than f_μ.

References

1. *Power Magnetics Design Optimization*, JUL19 revision, D. L. Feucht, Innovatia, innovatia.com
2. www.how2power.com/pdf_view.php?url=/newsletters/1504/H2PowerToday1504_FocusOnMagnetics.pdf

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